SPREADING FLOW OF AN AXISYMMETRIC LAMINAR JET OVER A HORIZONTAL OBSTACLE

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UDC 532.525.6

An approximate theory of an axisymmetric film flow with a hydraulic jump on a horizontal plate with a laminar liquid jet flowing onto it is proposed.

Properties of film flows generated by laminar and turbulent jets flowing onto solid surfaces are important for organizing the efficient cooling of heated bodies, and also for certain processes of physico-chemical treatment of materials. Such flows are distinguished by the specific feature, which is, as a rule, a region where a hydraulic jump appears, within which the film thickness can increase several-fold.

The theory of film flow of an identified type has been considered in [1-3], where field velocities in self-similar regions are described in detail and the relation is studied between the liquid layer thicknesses ahead of and beyond the hydraulic jump which follows from the requirement of the conservation of momentum in the integral form. However, the real physical reason which causes appearance of the jump remains absolutely unclear and additional relationships which enable one to finally determine all its characteristics have not been obtained. This problem has been solved for a film flow generated by a plane jet in [4], which reports that the appearence of the jump is caused by the necessity of the transition from a thin-layer flow in the region ahead of the jump to a quiet gravitational spreading flow of the liquid layer and the liquid draining from the area adjacent to the edge of the plate.

Below, an approximate theory of an axisymmetric film flow with a hydraulic jump on the horizontal plate with a laminar jet impinging on it is presented.

Outside the jet deflection region directly adjoining the impingence zone we can consider the film flow in the thin liquid layer approximation [1-4]. Here, before the hydraulic jump, two main regions should be distinguished: before the boundary layer (increasing downstream) joins the free surface of the film, where the total flux is composed of a viscous boundary layer flow near the plate surface and of a perfect flow outside of it, and the region after the junction, where the flow becomes viscous over the entire depth of the liquid layer. In both cases we have boundary layer equations in the form

$$u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}=\nu\frac{\partial^2 u_r}{\partial z^2}-\frac{1}{\rho}\frac{\partial p}{\partial r},\quad -\frac{1}{\rho}\frac{\partial p}{\partial z}-g=0,$$
$$\frac{\partial (ru_r)}{\partial r}+\frac{\partial (ru_z)}{\partial z}=0. \tag{1}$$

No-slip conditions are prescribed on the plate surface, it is required that $\partial u_x/\partial z = 0$ and $p = p_0 = \text{const}$ on the free surface after its junction with the boundary layer, and the usual conditions of boundary layer theory are formulated before the junction.

Neglecting gravity, self-similar velocity profiles in both regions under consideration have been found on the basis of the solution of the problem, formulated in [1, 2]. The approximate representations for them obtained by means of the Shvets method are presented in [3]. Exact profiles are expressed in terms of complicated functions of the dimensionless vertical coordinate, and approximate profiles taken from [3] are, altogether, not accurate enough for solving the problems of heat and

A. M. Gor'kii Ural State University, Ekaterinburg. Moscow Institute of Chemical Mechanical Engineering. Translated from Inzhenerno-fizicheskii Zhurnal, Vol. 64, No. 1, pp. 39-45, January, 1993. Original article submitted December 10, 1991.

mass transfer in a liquid layer. Therefore, bearing in mind the subsequent applications to heat and mass problems, we use here, as in [4], the integral method of Karman-Pohlhausen, presenting the velocity in the form

$$u_r(r,\ z) = U(r) f(\eta), \quad f = \frac{3}{2}(\eta - \eta^3/3), \quad \eta = z/\delta(r), \tag{2}$$

where $U(r) = U_0$, $\delta(r) < h(r)$ in the region prior to emergence of the boundary layer onto the free surface and $U(r) < U_0$, $\delta(t) = h(r)$ in the region of a viscous flow of the entire film. Profile (2) satisfies the imposed boundary conditions.

Integrating the first equation of (1) with allowance for the other two and formula (2) over dz from zero to $\delta(r) < h(r)$, we obtain

$$\frac{39}{280} U_0^2 \left(\frac{d}{dr} + \frac{1}{r}\right) \delta = \frac{3}{2} \frac{\nu U_0}{\delta} + g\delta \frac{dh}{dr}, \tag{3}$$

and the relation between $\delta(r)$ and $h(r)$ is defined from the condition of conservation of the fluid flow rate and is of the form

$$h(r) = a^2/2r + 3\delta(r)/8. \tag{4}$$

In the region of a viscous flow of the entire film, instead of (3), with allowance for conservation of the flow rate, after calculations we have

$$\frac{78}{875} a^4 U_0^2 \left(\frac{d}{dr} + \frac{1}{r}\right) \frac{1}{r^2 h} + \frac{2}{5} \frac{a^4}{r^3} \frac{U_0^2}{h^2} \frac{d(rh)}{dr} = \frac{6}{5} \frac{\nu a^2 U_0}{rh^2} + gh \frac{dh}{dr}, \tag{5}$$

here

$$U(r) = 4a^2 U_0/5rh. \tag{6}$$

It would be natural to introduce the dimensionless variables and parameters

$$\{\xi,\ \Delta,\ H\} = \frac{\mathrm{Re}^{1/3}}{a}\{r\,\mathrm{Re}^{-2/3},\ \delta,\ h\}, \quad \mathrm{Re} = \frac{U_0 a}{\nu}, \quad \mathrm{Fr} = \frac{ga}{U_0^2}. \tag{7}$$

Then from (3) and (4) we obtain the equation for $\Delta(\xi)$ in the form

$$\frac{39}{280} \frac{d}{d\xi}(\xi\Delta)^2 = 3\xi^2 + \mathrm{Re}^{-1/3}\mathrm{Fr}\left(\frac{3}{4}\xi^2 \frac{d\Delta}{d\xi} - 1\right)\Delta^2, \tag{8}$$

the solution of which, neglecting gravity (Fr = 0), is

$$\Delta = (280/39\xi)^{1/2} \approx 2.68\xi^{-1/2}. \tag{9}$$

The coordinate of the boundary layer emergence onto the free surface is defined from the condition $\Delta = H$, which yields

$$\xi_1 = \left(\frac{78}{875}\right)^{1/3} \approx 0.4467, \quad H_1 = H(\xi_1) = \frac{4}{5\xi_1} \approx 1.791. \tag{10}$$

By analogy, from (5) in the variables of (7), we get the equation

$$\left(\frac{272}{875} - \mathrm{Re}^{-1/3}\mathrm{Fr}\xi^2 H^3\right) \xi \frac{dH}{d\xi} = \frac{6}{5}\xi^2 - \frac{272}{875} H, \tag{11}$$

the solution of which at Fr = 0 is

$$H = \frac{175}{136}\xi^2 - \frac{C}{\xi}, \quad C = \frac{175}{136}\xi_1^3 - \Delta(\xi_1)\xi_1, \tag{12}$$

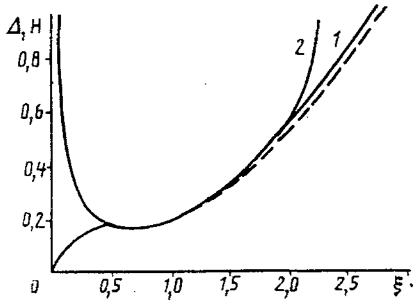

Fig. 1

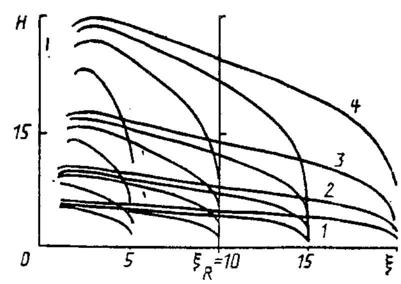

Fig. 2

Fig. 1. Dependence of Δ and H on ξ.

Fig. 2. Dependence of H on ξ for different values of ξ_R (figures on the ξ axis) and for the values of the parameter $\log(\mathrm{Re}^{-1/3}\mathrm{Fr}) = -2$ (1), -3 (2), -4 (3), -5 (4).

the integration constant being found under the condition that the value of H equals the one in (10) at $\xi = \xi_1$. The velocity at the free surface is expressed by equation (6).

As it follows from (11), the derivative $dH/d\xi$ formally tends to infinity when $\xi \to \xi_\infty - 0$, where ξ_∞ is a root of the equation $\xi^2_\infty H^3(\xi_\infty) = 272/875\mathrm{Re}^{-1/3}\mathrm{Fr}$, i.e., in a fairly extended film flow the film thickness inevitably grows over a certain area, and the validity of Eqs. (1) does not hold in it. This crisis is related to an increase in the hydrostatic pressure immediately near the solid surface which is caused by gravity in the film with increasing thickness. At $\xi = \xi_\infty$ and $z = 0$ the condition $\partial u_r/\partial z = 0$ is satisfied, similar to that at the point of detachment of a laminar boundary layer from the streamlined surface, indicating that a return flow appears at the plate surface. Such a behavior of the liquid layer is in agreement with the concept of Taney and Kuihary mentioned in [2] presenting a hydraulic jump as a result of the film detachment due to the appearance of a pressure gradient caused by gravity which retards the flow.

However, the computed values of ξ_∞ are found to be unreasonably large, and experiments show that the hydraulic jump starts at $\xi = \xi_- << \xi_\infty$. Moreover, as follows from the analysis reported in [2], in the region ahead of the jump $H \ll H(\xi_\infty)$ and the effect of gravity can be neglected. In Fig. 1 the approximate film profiles obtained from (9) and (12) at $\mathrm{Fr} = 0$ and by numerical integration of Eqs. (8) and (11) by the Runge–Kutta method of the 4th order at $\mathrm{Re}^{-1/3}\mathrm{Fr} = 10^{-4}$ (solid curves 1 and 2, respectively) and the profile which follows from a self-similar solution of the problem in [2] neglecting gravity (dashed line) are compared. First, it is clear that the fact that the Froude number is nonzero can be significant only in the region where the dimensionless thickness of the film is close to $H(\xi_\infty)$ which usually is not realized in experiments. And second, that the employment of approximation (2) for the velocity profile leads to an error not exceeding 5% over the whole area ahead of the jump. Analogous results follow also from the predictions for other fairly small values of $\mathrm{Re}^{-1/3}\mathrm{Fr}$. Therefore, for simplicity in subsequent calculations, the effect of gravity in the region under consideration is neglected. In the case of a plane film, the equations for dimensionless thicknesses of the boundary layer and the film itself, which are substituted for (8) and (11), can be integrated analytically [4].

In the region beyond the jump $\xi_+ < \xi < \xi_R$, where $\xi_R = (R/a)\mathrm{Re}^{-1/3}$ and $\xi_+ > \xi_-$, the viscous flow of a liquid layer where the thickness slowly varies with an increase in the radial coordinate can be described with the aid of equation (11) as was done previously. The boundary condition of "smooth draining" [3] from the edge of the plate should be presented in the form

$$dH/d\xi \to -\infty, \quad \xi \to \xi_R. \tag{13}$$

This condition is satisfied, if

$$H(\xi_R) = H_R = (875/272\,\mathrm{Re}^{-1/3}\mathrm{Fr}\,\xi_R^2)^{-1/3} \approx 0.68\,(a/R)^{2/3}\mathrm{Re}^{1/3}\mathrm{Fr}^{-1/3}, \tag{14}$$

and in the entire region under consideration $H(\xi)$ is a monotonically decreasing function. The latter means that in this region, the spreading flow of a liquid layer really occurs under gravity, which obviously cannot be neglected in Eq. (11). The characteristic dependences of $H(\xi)$ are presented in Fig. 2.

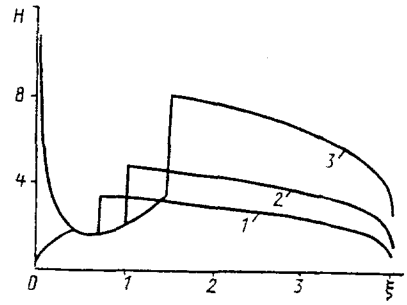

Fig. 3. Profiles of the liquid layer H(ξ) at $\xi_R = 4$ and $\mathrm{Re}^{-1/3}\mathrm{Fr} = 0.05$ (1), 0.01 (2), 0.001 (3).

Thus, within a certain interval $\xi_- < \xi < \xi_+$ the hydraulic jump is found to be necessary for the possibility of the transition from the thin film flow determined by the character of the onflowing jet to the regime of a gravitational spreading flow relative to the thick liquid layer, which assumes the condition of draining in form (13).

Next, considering the difference $\xi_+ - \xi_-$ to be relatively small, i.e., actually assuming $\xi_- \approx \xi_+ \approx \xi_j$, we neglect the hydrodynamic friction in the narrow region of the jump existence, which makes it possible to apply, as in [2-4], the condition of conservation of momentum in the form

$$\frac{1}{2} g (h_+^2 - h_-^2) \approx \int_0^{h_-} u_r^2 dz - \int_0^{h_+} u_r^2 dz, \tag{15}$$

wherein h_- and h_+ are dimensional thicknesses of the film before and after the jump near it. Using the dimensionless variables introduced above, we obtain an equation for determining the dimensionless coordinate ξ_j of the jump

$$\frac{1}{2} \mathrm{Re}^{-1/3}\mathrm{Fr} (H_+^2 - H_-^2) = \frac{272}{875\xi_j^2} \left(\frac{1}{H_-} - \frac{1}{H_+} \right), \tag{16}$$

where $H_- = H(\xi_j - 0)$ and $H_+ = H(\xi_j + 0)$ follow from the solution of Eq. (11) in the regions ahead of and beyond the hydraulic jump. It can be shown that Eq. 16) either has a single physically applicable solution, or has none at all. Characteristic profiles of the liquid layer at $\xi_R = 4$ and $\mathrm{Re}^{-1/3}\mathrm{Fr} = 0.05$, 0.01 and 0.001 are presented in Fig. 3. Dependences of the dimensionless jump radius and the quanties H_+ and H_- on ξ_R and parameter $\mathrm{Re}^{-1/3}\mathrm{Fr}$ are shown in Fig. 4.

The root of Eq. (16) is absent when ξ_R is smaller than certain minimum and larger than certain maximum values depending on $\mathrm{R}^{-1/3}\mathrm{Fr}$. Accordingly, the curves in Fig. 4a are cut off at the points corresponding to the extreme values. If $\xi_R < \xi_{min}$, what corresponds to small plates, the film detaches from the edge without the formation of a hydraulic jump and the flow over the entire region is of a dynamic thin-layer nature. On the contrary, if $\xi_R > \xi_{max}$, which corresponds to large plates, then the flow over the entire region is implemented in the regime of gravitational spreading, the initial thickness of the film being larger, the larger the plate radius.

To understand the way the properties of a hydraulic jump depend on the prescribed parameters of the flow, one should consider (besides the curves in Figs. 3, 4) the variation of these properties with an increase in the plate size, other conditions being equal. When the radius reaches a certain critical value, a jump develops for the first time at the plate edge of the film. With a further increase of radius, this jump gradually moves towards the incident jet and its intensity grows (i.e., H_+ monotonically increases, and H_- monotonically decreases). Finally, when the radius reaches the next critical value, the jump actually merges with the oncoming jet, and for radius values exceeding this critical one, the profile of the liquid layer is described by the curves like that illustrated in Fig. 2. This all can be clearly seen in Fig. 4a.

From the physical point of view, the described behavior of the hydraulic jump is determined by an increase in the total hydraulic resistance to the flow over the entire region up to the draining of the liquid. The same behavior must be also observed in the case when the resistance increase is caused not by an increase in the plate size, but by some other reasons. For example, it is possible to increase the resistance over the fixed-radius plate by passing over from the shape of the plate which provides "smooth" draining, to the sharp edge, or due to displacement of special rims near the edge, as was done in [3]. As follows from the analysis performed, any of these reasons can lead to the jump displacement towards the plate center,

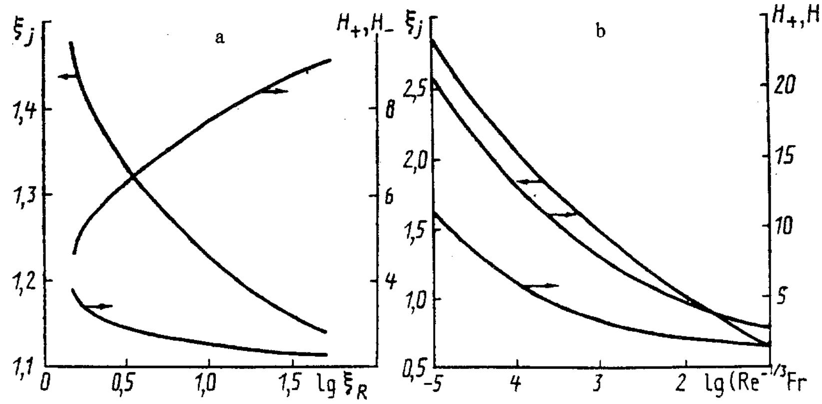

Fig. 4. Dependence of the dimensionless radius of the jump ξ_i and H_+, H_- on $\xi_R = (R/a)\mathrm{Re}^{-1/3}$ at $\mathrm{Re}^{-1/3}\mathrm{Fr} = 0$ (a) and the parameter $\mathrm{Re}^{-1/3}\mathrm{Fr}$ at $\xi_R = 4$ (b).

to the increase in a liquid layer thickness beyond the jump and its decrease ahead of the jump, which is fully in qualitative agreement with the experimental results [3]. Similar behavior of the jump, but under unsteady-state conditions, can also be observed while the vessel is filling up, with a jet incident on its bottom, which can be easily confirmed by placing a saucer under the water faucet jet.

A quantitative comparison of the theory with experimental data is hampered by the fact that the data reported in the known works are far from being enough for such a comparison. In Fig. 5, nevertheless, the theoretical dependence of w on x, where

$$w = \frac{\mathrm{Re}^{-1/3}\mathrm{Fr}}{\pi^2}\,\xi_j H_+^2 + \frac{1}{2\pi^2\xi_j H_+}, \qquad x = \frac{\xi_j}{\pi^{1/3}} \tag{17}$$

are dimensionless quantities introduced by Watson [2], is compared to the experimental results reported in [2] and [3]. An agreement between the theory and experiments may be considered altogether satisfactory. A certain systematic discrepancy between the theoretical results and most of the experimental data can evidently be attributed to the fact that in the theory the finiteness of the jump region width is neglected and self-similar profiles are spread over the parts separating different flow regions, in which self-similarity should be certainly lost.

The theory developed, certainly, does not hold in the region of jet deviation $r \lesssim r_0$, where the film flow just begins to develop. Since this very region is found to be very important for heat and mass applications, let us perform an approximate correction of the velocity field over this region near the plate surface. Assume, for simplicity, that for $r > r_0$ a self-similar flow of the type above develops and for $r < r_0$ the flow is established with an axisymmetric boundary layer described with the help of the known Blasius series [5]. Using the solution of the problem of a perfect fluid axisymmetric jet on flowing onto the plane plate, let us write down the relation for the velocity components near the onflow point [6]

$$u_r = 0.44U_0 r/a, \quad u_z = -0.88U_0 z/a.$$

These equations make it possible to determine the first term of the Blasius series for the radial velocity component in the boundary layer immediately near the plate surface in the form

$$u_r = 0.9277\cdot 0.44\sqrt{0.88\,\mathrm{Re}}\,U_0 rz/a^2 \approx 0.383\sqrt{\mathrm{Re}}\,U_0 rz/a^2. \tag{18}$$

The value of r_0 can be easily estimated when coincidence is required of (18) with the correspondent expression for u_r which follows from (2). After simple calculations, it yields

$$r_0 = (0.468)^{1/3}a \approx 0.776a. \tag{19}$$

In conclusion, let us briefly discuss the behavior of the heat- and mass-transfer rate along the surface of the plate in the flow. The analysis of transfer processes in a liquid layer is a fairly complicated independent problem [7]. Qualitative information, however, can be obtained by considering the limiting case of fairly large Peclet numbers, when the known

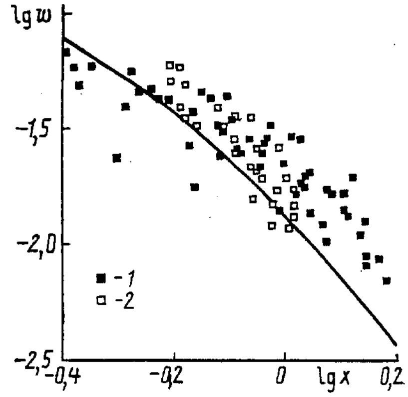

Fig. 5. Comparison of the relation w = f(x) in accordance with (17) (the curve) with experimental data reported in [2] and [3] (points 1 and 2, respectively).

(or Sherwood) number have been obtained in [3, 4] and in a number of other works; within the limits of the flow self-similarity regions this number decreases inversely proportionally to the square root of r.

If the self-similar velocity profile of the thin-layer flow is extended over to the region of the jet deviation up to the point of onflow, one can come to the absurd conclusion that it tends to infinity in the vicinity of the indicated point [3, 7]. Actually, in this vicinity the velocity of an ideal fluid at the outer boundary of the hydrodynamic boundary layer is not constant, but quickly increases with r from zero up to U_0. Taking this into account, one comes to the conclusion that within the range of the region with a linear size (1), the Nusselt number is approximately constant, which agrees with the experimental data reported in [8]. An analogous conclusion has been also made for a plane film flow in [4].

To get an idea about the character of the Nusselt number variation over the region of a hydraulic jump one should take into account the fact that in this region a vortex motion actually develops, forcing back the liquid from the surface and providing its mixing. This conforms with the known model of surface restoration, in accordance with which after the hydraulic jump a thermal or diffusional boundary layer again starts to grow from zero. Correspondingly, the Nusselt or Sherwood number over the region of gravitational spreading flow decreases by the same law as over the region of the thin-layer flow before the jump. At the same time, its initial value can be higher than one, obtained over the final region immediately ahead of the jump. Thus, we come to a conclusion about the presence of the maximum intensity of heat and mass transfer over the region of a hydraulic jump, which agrees with the observations [3] and corresponds to the analysis reported in [4].

NOTATION

a, radius of the onflowing jet from the plate; g, free-fall acceleration; h, H, dimensional and dimensionless thickness of the spreading film; R, radius of the plate; p, pressure; U_0, velocity of the jet onflow; u_r and u_z, velocity components; r and z, radial and vertical coordinates; δ and Δ, dimensional and dimensionless thicknesses of the boundary layer; ξ, dimensionless radial coordinate; η, dimensionless vertical coordinate; Re and Fr, Reynolds and Froude numbers.

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EFFECT OF THE PERFORATION OF THE PIPELINE WALL ON THE VELOCITY OF PROPAGATION OF LONG COMPRESSION WAVES IN A FLUID

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UDC 539.4.015.532.59

A model for predicting the velocity of a weak compression wave propagating over a fluid in a tube with perforated holes is suggested. The fluid is retained in the perforations by surface tension. The wave velocity weakly decreases with an increase in the pressure drop over the wave. Examples of particular predictions are given.

The sound wall and shock wave propagation over the fluid in tubes has specific features which are of interest from both theoretical and practical points of view. The speed of sound in such systems can be fairly different from the case of an infinite medium. The dependence of the shock wave speed on the pressure can be also quite peculiar.

The speed of sound C for fluids in tubes has been calculated by N. E. Zhukovskii [1] and is defined by the equation

$$C^2 = \frac{C_f^2}{1 + 2r_0\rho_0 C_f^2/lE}, \tag{1}$$

where E is the Young modulus for the material of tube walls; r_0 and ℓ are the radius of the tube and the thickness of its walls, respectively (it is assumed that $l \ll r_0$).

We next consider sound waves and finite-amplitude waves in the fluid that occupy the tube with perforated walls. It is assumed that the fluid does not wet an external surface of the tube. All perforations are of the same size (diameter A), and on the average, are uniformly distributed over the tube surface. It is assumed that the tube is of an infinite length, and that the wave length is much larger than other linear sizes in the problem [the tube diameter and average distance (A + B) between the perforations]. At the same time, the period when the fluid is present in the wave is large in comparison with the time of establishing a static condition of the fluid in perforation holes. Capillary forces prevent the fluid from effluxing out of the holes. The maximum increment in pressure P^* in the wave at which the capillary forces still retain the fluid in the holes (see Fig. 1) is equal to $P^* = 4\sigma/A$. The gravity is not taken into account. This is admissible in the case of a tube in a horizontal position in the gravity field and when the inequality $\sigma >> \rho_0 g A r_0$ holds.

A plane weak shock wave of infinite length propagating over the fluid in the tube (with no allowance for the friction against the walls) is completely determined by the equations of conservation of mass, momentum, and isentropic compression of a fluid

$$\rho_0 U_0 (S_0 + S_A) = \rho_1 U_1 (S_0 + \Delta S_0) + \rho_1 U_0 S_A; \tag{2}$$

$$(P_0 + \rho_0 U_0^2)(S_0 + S_A) = (P_1 + \rho_1 U_1^2)(S_0 + \Delta S_0) + (P_1 + \rho_1 U_0^2)(S_A + S_h); \tag{3}$$

$$P = P(\rho), \; P = P_1 - P_0, \; \rho = \rho_1 - \rho_0, \tag{4}$$

N. N. Semenov Moscow Institute of Chemical Physics of the Russian Academy of Sciences. Translated from Inzhenerno-fizicheskii Zhurnal, Vol. 64, No. 1, pp. 46-49, January, 1993. Original article submitted December 9, 1991.